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Multiple travelling wave solutions of nonlinear evolution equations using a unified algebraic method

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Abstract
A new direct and unified algebraic method for constructing multiple travelling wave solutions of general nonlinear evolution equations is presented and implemented in a computer algebraic system. Compared with most of the existing tanh methods, the Jacobi elliptic function method or other sophisticated methods, the proposed method not only gives new and more general solutions, but also provides a guideline to classify the various types of the travelling wave solutions according to the values of some parameters. The solutions obtained in this paper include (a) kink-shaped and bell-shaped soliton solutions, (b) rational solutions, (c) triangular periodic solutions and (d) Jacobi and Weierstrass doubly periodic wave solutions. Among them, the Jacobi elliptic periodic wave solutions exactly degenerate to the soliton solutions at a certain limit condition. The efficiency of the method can be demonstrated on a large variety of nonlinear evolution equations such as those considered in this paper, KdV–MKdV, Ito's fifth MKdV, Hirota, Nizhnik–Novikov–Veselov, Broer–Kaup, generalized coupled Hirota–Satsuma, coupled Schrödinger–KdV, $(2+1)$-dimensional dispersive long wave, $(2+1)$-dimensional Davey–Stewartson equations. In addition, as an illustrative sample, the properties of the soliton solutions and Jacobi doubly periodic solutions for the Hirota equation are shown by some figures. The links among our proposed method, the tanh method, extended tanh method and the Jacobi elliptic function method are clarified generally.

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1. Introduction

The investigation of the travelling wave solutions of nonlinear evolution equations plays an important role in the study of nonlinear wave phenomena. The wave phenomena observed in fluid dynamics, plasma, elastic media, optical fibres, etc are often modelled by bell-shaped sech solutions and kink-shaped tanh solutions. The analytical solutions, if available, of nonlinear evolution equations facilitate the verification of numerical solvers and aids in the stability analysis of solutions. In the past decades, both mathematicians and physicists have made significant progression in this direction. Many effective methods such as the inverse scattering method [1, 2], Darboux transformation [3–10], the Hirota bilinear method [11–13], the homogeneous balance method [14–16] and the tanh method [17–22] have been developed. Among those, the tanh method is considered to be the most effective and direct algebraic method for solving nonlinear equations. In recent years, much work has been concentrated on the various extensions and applications of the tanh method. The basic purpose of these papers is to simplify the routine calculation of the method or obtain more general solutions of nonlinear evolution equations [18–28]. The Weierstrass and Theta elliptic functions can be used to find periodic solutions by applying spectral theory for KdV equation, coupled nonlinear Schrödinger equation, etc. But usually this method is applied to the integrable nonlinear evolution equations admitting Lax pairs representation [29, 30]. An alternative method is to transform the equation under study to the Weierstrass equation, Jacobi equation, or more generally, to Painlevé-type equations [31, 32]. This procedure is, in general, complicated or impossible, especially for dissipative nonlinear evolution equations and nonlinear coupled equations. Very recently, a Jacobi elliptic function expansion method, which is straightforward and effective, was proposed for constructing periodic wave solutions for some nonlinear evolution equations [33–35]. The essential idea of this method is similar to the tanh method by replacing the tanh function with some Jacobi elliptic functions such as $\mathrm{sn}\,\xi$, $\mathrm{cn}\,\xi$ and $\mathrm{dn}\,\xi$. For example, the Jacobi periodic solution in terms of $\mathrm{sn}\,\xi$ may be obtained by applying the sn-function expansion. Many similarly repetitious calculations have to be done to search for the Jacobi doubly periodic wave solutions in terms of $\mathrm{cn}\,\xi$ and $\mathrm{dn}\,\xi$. If the equation does not admit such types of solutions, these efforts will be in vain.

In this paper, we shall develop a new algebraic method with symbolic computation for obtaining the above-mentioned various travelling wave solutions in a unified way. Compared with most of the existing methods, the proposed method not only gives a unified formulation to construct various travelling wave solutions, but also provides a guideline to classify the various types of the travelling wave solutions according to the values of some parameters. The paper is arranged as follows. In section 2, we describe our proposed method in detail and explain its links with the tanh method, the extended tanh method and Jacobi elliptic function method. In section 3, the proposed method is applied to various nonlinear evolution equations for constructing new and fundamental multiple travelling wave solutions including soliton, rational, triangular periodic, Jacobi and Weierstrass doubly periodic solutions. A conclusion is then given in section 4.

2. Methodology

We shall describe the motivation of our method starting from the description of the tanh method [17–22]. It is well known that for a given nonlinear equation

$$H(u, u_t, u_x, u_{xx}, \ldots) = 0 \tag{2.1}$$

its travelling wave soliton solution can often be expressed as a polynomial of tanh function, namely,

$$u(x,t) = U(\xi) = \sum_{i=0}^{n} a_i \varphi^i \tag{2.2}$$

where $\varphi = \tanh\xi$ with $\xi = x + ct$. The positive integer n can be determined by balancing the highest derivative term with the nonlinear terms in equation (2.1). We substitute expansion (2.2) into equation (2.1), and then equate to zero the coefficients of φ^i to obtain the explicit expressions for $c, a_0, \ldots, a_n$. Recently, by taking advantage of the property that the Riccati equation

$$\varphi' = b + \varphi^2 \tag{2.3}$$

admits several types of solutions, we proposed an extended tanh method which can be used to obtain more general travelling wave solutions than the tanh method. The key idea of the extended tanh method is to replace the $\tanh\xi$ in expansion (2.2) by the solution of the Riccati equation (2.3) [25, 26].

Now we try to generalize equation (2.3) so that the more general solutions of physical relevance can be found. Then the following observations may be helpful to us. First, often physically localized soliton solutions of most nonlinear evolution equations are the superposition and/or combinations of different powers of $\tanh\xi$ and $\operatorname{sech}\xi$, which exactly solve the following equations:

$$\varphi' = \sqrt{1 - 2\varphi^2 + \varphi^4} \tag{2.4}$$

and

$$\varphi' = -\varphi\sqrt{1 - \varphi^2} \tag{2.5}$$

respectively. On the other hand, there are two standard forms of elliptic functions in the special function theory. One is the Weierstrass elliptic function $\wp(\xi, g_2, g_3)$ which is doubly periodical and satisfies the equation

$$\varphi' = \pm\sqrt{-g_3 - g_2\varphi + 4\varphi^3} \tag{2.6}$$

where g_2, g_3 are called invariants of the Weierstrass elliptic function. The other is the Jacobi elliptic function $\operatorname{sn}\xi = \operatorname{sn}(\xi|m)$, which satisfies the equation

$$\varphi' = \pm\sqrt{1 - (m^2 + 1)\varphi^2 + m^2\varphi^4} \tag{2.7}$$

where m is a modulus of Jacobi elliptic functions. Jacobi elliptic functions are doubly periodical and possess properties of triangular functions:

$$\operatorname{sn}^2\xi + \operatorname{cn}^2\xi = 1 \qquad \operatorname{dn}^2\xi = 1 - m^2\operatorname{sn}^2\xi$$
$$(\operatorname{sn}\xi)' = \operatorname{cn}\xi\operatorname{dn}\xi \qquad (\operatorname{cn}\xi)' = -\operatorname{sn}\xi\operatorname{dn}\xi \qquad (\operatorname{dn}\xi)' = -m^2\operatorname{sn}\xi\operatorname{cn}\xi.$$

When $m \to 1$, the Jacobi functions degenerate to the hyperbolic functions, i.e.

$$\operatorname{sn}\xi \to \tanh\xi \qquad \operatorname{cn}\xi \to \operatorname{sech}\xi \qquad \operatorname{dn}\xi \to \operatorname{sech}\xi$$

and when $m \to 0$, the Jacobi functions degenerate to the triangular functions, i.e.

$$\operatorname{sn}(\xi, m) \to \sin\xi \qquad \operatorname{cn}\xi \to \cos\xi \qquad \operatorname{dn}\xi \to 1.$$

The more detailed notations for the Weierstrass and Jacobi elliptic functions can be found in [36, 37]. Now replacing equation (2.3) by equation (2.6) or equation (2.7), we can obtain the Jacobi and Weierstrass elliptic doubly periodic solutions, if available, of a given nonlinear evolution equation.

The above hints and analysis motivate us to develop a direct and unified scheme for constructing a series of travelling wave solutions. For this purpose, we extend equations (2.4)–(2.7) to the following more general and unified equation:

$$\varphi' = \varepsilon\sqrt{\sum_{j=0}^{r} c_j\varphi^j} \tag{2.8}$$

where $\varepsilon = \pm 1$. The positive integer r and constants $c_0, c_1, \ldots, c_r$ are to be determined.

We remark here that the proposed method contains two balance parameters n and r. In general, balancing the highest derivative term with nonlinear terms in equation (2.1) leads to a special relation for n and r. For example, in the case of the MKdV equation

$$u_t + 6u^2 u_x + u_{xxx} = 0$$

we have

$$r = 2(n+1) \tag{2.9}$$

and in the case of the Kawachara equation

$$u_t + uu_x + \alpha u_{xxx} + \beta u_{xxxxx} = 0 \tag{2.10}$$

we have

$$n = 2(r-2). \tag{2.11}$$

Relations (2.9) and (2.11) will give the different possible values of n and r, which then lead to the series expansions of the exact solutions for the above equations. For further illustration, if we take $n = 2$ and $r = 3$ in (2.11), we obtain the following series expansion of an exact solution for the Kawachara equation (2.10):

$$u = a_0 + a_1\varphi + a_2\varphi^2 \qquad \varphi' = \varepsilon\sqrt{c_0 + c_1\varphi + c_2\varphi^2 + c_3\varphi^3}.$$

Similarly, taking $n = 4$, $r = 4$ in (2.11), we have

$$u = a_0 + a_1\varphi + a_2\varphi^2 + a_3\varphi^3 + a_4\varphi^4 \qquad \varphi' = \varepsilon\sqrt{c_0 + c_1\varphi + c_2\varphi^2 + c_3\varphi^3 + c_4\varphi^4}.$$

We see that the travelling wave solutions of equation (2.1) depend on the explicit solvability of (2.8) with its coefficients c, a_i, c_j satisfying a system of algebraic equations. The solution of such a system will be getting tedious with the increase of the values of n and r. But in the case of equation (3.1) when $r = 4$, that is, equation (2.8) gives a series of fundamental solutions such as soliton, rational, triangular periodic, Jacobi and Weierstrass doubly periodic solutions. We only consider the case $r = 4$ in this paper and hence

$$\varphi' = \varepsilon\sqrt{c_0 + c_1\varphi + c_2\varphi^2 + c_3\varphi^3 + c_4\varphi^4}. \tag{2.12}$$

Theorem 1. *Suppose that φ is a solution of equation (2.12), then we have the following results:*

(i) If $c_3 = c_1 = c_0 = 0$, equation (2.12) possesses a bell-shaped soliton solution

$$\varphi = \sqrt{-\frac{c_2}{c_4}}\,\mathrm{sech}(\sqrt{c_2}\xi) \qquad c_2 > 0 \quad c_4 < 0 \tag{2.13}$$

a triangular solution

$$\varphi = \sqrt{-\frac{c_2}{c_4}}\,\sec(\sqrt{-c_2}\xi) \qquad c_2 < 0 \quad c_4 > 0 \tag{2.14}$$

and a rational solution

$$\varphi = -\frac{\varepsilon}{\sqrt{c_4}\xi} \qquad c_2 = 0 \quad c_4 > 0. \tag{2.15}$$

(ii) If $c_3 = c_1 = 0$, $c_0 = \frac{c_2^2}{4c_4}$, *equation (2.12) possesses a kink-shaped soliton solution*

$$\varphi = \varepsilon\sqrt{-\frac{c_2}{2c_4}}\tanh\left(\sqrt{-\frac{c_2}{2}}\xi\right) \qquad c_2 < 0 \quad c_4 > 0 \tag{2.16}$$

and a triangular solution

$$\varphi = \varepsilon\sqrt{\frac{c_2}{c_4}}\tan\left(\sqrt{\frac{c_2}{2}}\xi\right) \qquad c_2 > 0 \quad c_4 > 0. \tag{2.17}$$

(iii) If $c_3 = c_1 = 0$, *equation (2.12) admits three Jacobi elliptic function solutions*

$$\varphi = \sqrt{\frac{-c_2m^2}{c_4(2m^2-1)}}\,\mathrm{cn}\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right) \qquad c_2 > 0 \quad c_0 = \frac{c_2^2m^2(m^2-1)}{c_4(2m^2-1)^2} \tag{2.18}$$

$$\varphi = \sqrt{\frac{-c_2}{c_4(2-m^2)}}\,\mathrm{dn}\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right) \qquad c_2 > 0 \quad c_0 = \frac{c_2^2(1-m^2)}{c_4(2-m^2)^2} \tag{2.19}$$

and

$$\varphi = \varepsilon\sqrt{\frac{-c_2m^2}{c_4(m^2+1)}}\,\mathrm{sn}\left(\sqrt{\frac{-c_2}{m^2+1}}\xi\right) \qquad c_2 < 0 \quad c_0 = \frac{c_2^2m^2}{c_4(m^2+1)^2}. \tag{2.20}$$

Here we generally clarify whether the modulus m *will appear in the Jacobi doubly periodic solutions (2.18)–(2.20) of equation (2.12). In the case when* $c_3 = c_1 = 0$, *by using the transformations*

$$c_0 = \frac{c_2^2m^2}{c_4(m^2+1)^2} \qquad \bar{\varphi} = \sqrt{-\frac{c_4(m^2+1)}{c_2m^2}}\varphi \qquad \bar{\xi} = \sqrt{-\frac{c_2}{m^2+1}}\xi$$

equation (2.12) is reduced to equation (2.7) and hence we can obtain the solution (2.20). Again by using the relations among functions $\mathrm{sn}\,\xi$, $\mathrm{cn}\,\xi$ *and* $\mathrm{dn}\,\xi$, *we also get solutions (2.18) and (2.19). As* $m \to 1$ *the Jacobi doubly periodic solutions (2.18) and (2.19) all degenerate to the soliton solutions (2.13), and the Jacobi doubly periodic solution (2.20) degenerates to (2.16).*

(iv) If $c_4 = c_1 = c_0 = 0$, *equation (2.12) possesses a bell-shaped soliton solution*

$$\varphi = -\frac{c_2}{c_3}\,\mathrm{sech}^2\left(\frac{\sqrt{c_2}}{2}\xi\right) \qquad c_2 > 0 \tag{2.21}$$

a triangular solution

$$\varphi = -\frac{c_2}{c_3}\sec^2\left(\frac{\sqrt{-c_2}}{2}\xi\right) \qquad c_2 < 0 \tag{2.22}$$

and a rational solution

$$\varphi = \frac{1}{c_3\xi^2} \qquad c_2 = 0. \tag{2.23}$$

(v) If $c_4 = c_2 = 0$, $c_3 > 0$, *equation (2.12) admits a Weierstrass elliptic function solution*

$$\varphi = \wp\left(\frac{\sqrt{c_3}}{2}\xi, g_2, g_3\right) \tag{2.24}$$

where $g_2 = -4c_1/c_3$ *and* $g_3 = -4c_0/c_3$ *are called invariants of Weierstrass elliptic function. Actually, in the case when* $c_2 = c_4 = 0$, *by using the transformations*

$$\bar{\xi} = \frac{\sqrt{c_3}}{2}\xi \qquad c_0 = -\frac{1}{4}c_3g_3 \qquad c_1 = -\frac{1}{4}c_3g_2$$

equation (2.12) is reduced to equation (2.6) and hence we have (2.24).

Remark 1. The other types of travelling wave solutions such as csc ξ, cot ξ, csch ξ and coth ξ can be obtained in theorem 1; they appear in pairs with the corresponding sec ξ, tan ξ, sech ξ and tanh ξ. Since there is no clear physical relevance on the singular solutions in terms of functions sec ξ, tan ξ and $1/\xi$, we do not consider these solutions in this paper.

Remark 2. Let us consider three special cases of our proposed method. In the case when $c_1 = c_3 = 0, c_0 = 1, c_2 = -2, c_4 = 1$, equation (2.12) has a solution tanh ξ and our method reduces to the tanh method [17, 18]. In the case when $c_1 = c_3 = 0, c_0 = b^2, c_2 = 2b, c_4 = 1$, equation (2.12) degenerates to the Riccati equation (2.3); our proposed method becomes the extended tanh method [26, 27]. The cases (2.18)–(2.20) readily cover the results of Jacobi function expansion method [33–35]. In conclusion, our proposed method is a generalization of the tanh method, the extended tanh method and Jacobi elliptic function method.

The algorithm presented here is also a computerizable method, in which generating an algebraic system from equation (2.1) and solving it are two key procedures and laborious to do by hand. But they can be implemented on a computer with the help of computer algebra software such as *Mathematica* or *Maple*. The output solutions from the algebraic system comprise a list of the form $\{c, a_i, c_j\}$. In general, if the values of some parameters are left unspecified, then they are regarded to be arbitrary in the solution of equation (2.1).

3. Applications

In this section, we apply the technique developed in section 2 to various nonlinear evolution equations which may be both integrable and non-integrable.

Example 1. Consider generalized coupled Hirota–Satsuma equation [11, 13]

$$\begin{aligned} u_t &= \tfrac{1}{4}u_{xxx} + 3uu_x + 3(v^2 + w)_x \\ v_t &= -\tfrac{1}{2}v_{xxx} - 3uv_x \\ w_t &= -\tfrac{1}{2}w_{xxx} - 3uw_x. \end{aligned} \tag{3.1}$$

This system was introduced by Satsuma and Hirota [11]. They found its three-soliton solutions and showed that the well-known Hirota–Satsuma coupled KdV equation is a special case of system (3.1) with $w = 0$ and $x \to \sqrt{2}x$, $t \to \sqrt{2}t$. Recently starting from its bilinear form, Tam *et al* revisited system (3.1) and found a new type of soliton solution [13]. Here our proposed method will give a series of travelling wave solutions for equation (3.1) as follows.

Using transformations $u(x, t) = U(\xi)$, $v(x, t) = V(\xi)$, $w(x, t) = W(\xi)$, $\xi = x + ct$, we reduce equation (3.1) to a system of ordinary differential equations:

$$\begin{aligned} cU' &= \tfrac{1}{4}U''' + 3UU' + 3(-V^2 + W)' = 0 \\ cV' &= -\tfrac{1}{2}V''' - 3UV' \\ cW' &= -\tfrac{1}{2}W''' - 3UW'. \end{aligned} \tag{3.2}$$

We expand the solution of equation in the form

$$U = \sum_{i=0}^{n_1} a_i\varphi^i \qquad V = \sum_{i=0}^{n_2} b_i\varphi^i \qquad W = \sum_{i=0}^{n_3} d_i\varphi^i$$

where φ satisfies (2.8).

Balancing the highest linear terms with nonlinear terms in (3.2), we obtain

$$r = n_1 + 2 \qquad n_2 \leqslant n_1 \qquad n_3 \leqslant 2n_1.$$

Therefore, we may choose $r = 4$, $n_1 = n_2 = n_3 = 2$ and have the following expansions:

$$U = a_0 + a_1\varphi + a_2\varphi^2 \qquad V = b_0 + b_1\varphi + b_2\varphi^2 \qquad W = d_0 + d_1\varphi + d_2\varphi^2 \tag{3.3}$$

where φ satisfies (2.12).

With the help of the symbolic software *Mathematica*, by substituting (3.3) into (3.2) and setting the coefficients of $\varphi^i\sqrt{c_0 + c_1\varphi + c_2\varphi^2 + c_3\varphi^3 + c_4\varphi^4}$ $(i = 0, 1, \ldots)$ to zero, we further obtain a system of algebraic equations:

$$4\varepsilon ca_1 - 12\varepsilon a_0a_1 + 24\varepsilon b_0b_1 - 3\varepsilon^3a_2c_1 - \varepsilon^3a_1c_2 - 12\varepsilon d_1 = 0$$
$$-12\varepsilon a_1^2 + 8\varepsilon ca_2 - 24\varepsilon a_0a_2 + 24\varepsilon b_1^2 + 48\varepsilon b_0b_2 - 8\varepsilon^3a_2c_2 - 3\varepsilon^3a_1c_3 - 24\varepsilon d_2 = 0$$
$$-12\varepsilon a_1a_2 + 24\varepsilon b_1b_2 - 5\varepsilon a_2c_3 - 2\varepsilon^3a_1c_4 = 0$$
$$-\varepsilon a_2^2 + 4\varepsilon b_2^2 - \varepsilon^3a_2c_4 = 0$$
$$2\varepsilon cb_1 + 6\varepsilon a_0b_1 + 3\varepsilon^3b_2c_1 + \varepsilon^3b_1c_2 = 0$$
$$6\varepsilon a_1b_1 + 4\varepsilon cb_2 + 12\varepsilon a_0b_2 + 8\varepsilon^3b_2c_2 + 3\varepsilon^3b_1c_3 = 0$$
$$6\varepsilon a_2b_1 + 12\varepsilon a_1b_2 + 15\varepsilon^3b_2c_3 + 6\varepsilon^3b_1c_4 = 0$$
$$\varepsilon a_2b_2 + 2\varepsilon^3b_2c_4 = 0$$
$$2\varepsilon cd_1 + 6\varepsilon a_0d_1 + \varepsilon^3c_2d_1 + 3\varepsilon^3c_1d_2 = 0$$
$$6\varepsilon a_1d_1 + 3\varepsilon^3c_3d_1 + 4\varepsilon cd_2 + 12\varepsilon a_0d_2 + 8\varepsilon^3c_2d_2 = 0$$
$$6\varepsilon a_2d_1 + 6\varepsilon^3c_4d_1 + 12\varepsilon a_1d_2 + 15\varepsilon^3c_3d_2 = 0$$
$$\varepsilon a_2d_2 + 2\varepsilon^3c_4d_2 = 0.$$

Note that $\varepsilon = \pm 1$ and hence $\varepsilon^3 = \varepsilon$. We may eliminate ε from the above system. From the output of *Mathematica*, we find three kinds of solutions, namely,

$$\begin{aligned} &c_3 = c_1 = a_1 = b_1 = d_1 = 0 \qquad a_0 = -\tfrac{1}{3}(c + 2c_2)\\ &c_4 = -\tfrac{1}{2}a_2 \qquad b_2 = \pm\tfrac{1}{2}a_2 \qquad d_2 = \tfrac{1}{3}a_2(2c \pm 3b_0 + c_2) \end{aligned} \tag{3.4}$$

with $a_2, b_0, d_0, c_0, c_2, c$ being arbitrary constants,

$$\begin{aligned} &c_4 = a_2 = b_2 = d_2 = 0 \qquad a_0 = -\tfrac{1}{6}(2c + c_2) \qquad c_3 = -2a_1\\ &b_1 = \pm\tfrac{1}{2}a_1 \qquad d_1 = \tfrac{1}{12}a_1(8c \pm 12b_0 + c_2) \end{aligned} \tag{3.5}$$

with $a_1, b_0, d_0, c_0, c_1, c_2, c$ being arbitrary constants and

$$\begin{aligned} &c_3 = a_1 = b_2 = d_2 = 0 \qquad a_0 = -\frac{ca_2 + b_1^2}{a_2} \qquad c_4 = -a_2\\ &d_1 = \frac{1}{8}(16b_0b_1 - 2a_2c_1) \qquad c = \frac{a_2c_2 - b_1^2}{4a_2} \end{aligned} \tag{3.6}$$

with $b_0, b_1, c_0, c_1, a_2 \neq 0$ being arbitrary constants.

Now all possible explicit solutions of the generalized coupled Hirota–Satsuma equation (3.1) are discussed as follows:

(A) From (2.13) and (3.4), we obtain a soliton solution as follows:

$$\begin{aligned} &u_1 = -\tfrac{1}{3}(c + 2c_2) + 2c_2\,\mathrm{sech}^2(\sqrt{c_2}\xi)\\ &v_1 = b_0 \pm c_2\,\mathrm{sech}^2(\sqrt{c_2}\xi)\\ &w_1 = d_0 + \tfrac{2}{3}(2c \pm 3b_0 + c_2)\,\mathrm{sech}^2(\sqrt{c_2}\xi) \qquad c_2 > 0 \end{aligned} \tag{3.7}$$

where $\xi = x + ct$. The solution obtained from (2.16) and (3.4) is the same as solution (3.7) by considering a simple transformation $c_2 \to -2c_2$.

Again from (2.18) and (3.4), we get a Jacobi doubly periodic solution

$$\begin{aligned} u_2 &= -\frac{1}{3}(c+2c_2) + \frac{2c_2m^2}{2m^2-1}\,\mathrm{cn}^2\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right) \\ v_2 &= b_0 \pm \frac{c_2m^2}{2m^2-1}\,\mathrm{cn}^2\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right) \\ w_2 &= d_0 + \frac{2}{3}(2c \pm 3b_0 + c_2)\frac{m^2}{2m^2-1}\,\mathrm{cn}^2\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right) \qquad c_2 > 0 \end{aligned} \tag{3.8}$$

where $\xi = x + ct$. From (2.19), (2.20) and (3.4), we also get two Jacobi doubly periodic solutions, but they belong to the same kind of solution as (3.8) under the following transformations:

$$c_2 \to \frac{c_2(2-m^2)}{2m^2-1} \qquad c_2 \to -\frac{c_2(m^2+1)}{2m^2-1}$$

respectively. As $m \to 1$, the Jacobi periodic solution (3.8) degenerates to the soliton solution (3.7). If we take $d_0 = 0, b_0 = \mp\frac{1}{3}(2c + c_2)$, then the solutions (3.7) and (3.8) directly lead to the solutions of the well-known Hirota–Satsuma coupled equation.

(B) (2.21) and (3.5) give the same solution as (3.7) by transformation $c_2 \to 4c_2, a_1 \to a_2$. Taking $c_2 = 0$ in (3.5) and then using (2.24), we obtain a Weierstrass periodic solution

$$\begin{aligned} u_3 &= -\frac{1}{3}c + a_1\wp\left(\sqrt{-\frac{a_1}{2}}\xi, g_2, g_3\right) \\ v_3 &= b_0 \pm \frac{1}{2}a_1\wp\left(\sqrt{-\frac{a_1}{2}}\xi, g_2, g_3\right) \\ w_3 &= d_0 + \frac{1}{3}(2c \pm 3b_0)a_1\wp\left(\sqrt{-\frac{a_1}{2}}\xi, g_2, g_3\right) \qquad a_1 < 0 \end{aligned}$$

where $g_2 = 2c_1/a_1, g_3 = 2c_0/a_1$ and $\xi = x + ct$. Taking $d_0 = 0, b_0 = \mp\frac{2}{3}c$, the Weierstrass periodic solution is exactly that of the Hirota–Satsuma coupled equation.

(C) Taking $c_1 = 0$ in (3.6) and then using (2.13) and (2.16), we obtain two soliton solutions

$$\begin{aligned} u_4 &= -\frac{ca_2 + b_1^2}{a_2} + c_2\,\mathrm{sech}^2(\sqrt{c_2}\xi) \\ v_4 &= b_0 + b_1\sqrt{\frac{c_2}{a_2}}\,\mathrm{sech}(\sqrt{c_2}\xi) \\ w_4 &= d_0 + 2b_0b_1\sqrt{\frac{c_2}{a_2}}\,\mathrm{sech}(\sqrt{c_2}\xi) \qquad c_2 > 0 \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} u_5 &= -\frac{ca_2 + b_1^2}{a_2} + \frac{1}{2}c_2\tanh^2\left(\sqrt{-\frac{c_2}{2}}\xi\right) \\ v_5 &= b_0 \pm b_1\sqrt{\frac{c_2}{2a_2}}\tanh\left(\sqrt{-\frac{c_2}{2}}\xi\right) \\ w_5 &= d_0 \pm 2b_0b_1\sqrt{\frac{c_2}{2a_2}}\tanh\left(\sqrt{-\frac{c_2}{2}}\xi\right) \qquad c_2 < 0 \end{aligned} \tag{3.10}$$

where $\xi = x + \frac{a_2c_2 - 6b_1^2}{4a_2}t$.

Again from (2.18)–(2.20), we find three Jacobi doubly periodic solutions

$$\begin{aligned} u_6 &= -\frac{ca_2 + b_1^2}{a_2} + \frac{m^2 c_2}{2m^2 - 1}\,\mathrm{cn}^2\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right) \\ v_6 &= b_0 + b_1\sqrt{\frac{m^2 c_2}{a_2(2m^2-1)}}\,\mathrm{cn}\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right) \\ w_6 &= d_0 + 2b_0 b_1\sqrt{\frac{m^2 c_2}{a_2(2m^2-1)}}\,\mathrm{cn}\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right) \qquad c_2 > 0 \end{aligned} \tag{3.11}$$

$$\begin{aligned} u_7 &= -\frac{ca_2 + b_1^2}{a_2} + \frac{c_2}{2 - m^2}\,\mathrm{dn}^2\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right) \\ v_7 &= b_0 + b_1\sqrt{\frac{c_2}{a_2(2-m^2)}}\,\mathrm{dn}\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right) \\ w_7 &= d_0 + 2b_0 b_1\sqrt{\frac{c_2}{a_2(2-m^2)}}\,\mathrm{dn}\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right) \qquad c_2 > 0 \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} u_8 &= -\frac{ca_2 + b_1^2}{a_2} + \frac{m^2 c_2}{m^2 + 1}\,\mathrm{sn}^2\left(\sqrt{-\frac{c_2}{m^2+1}}\xi\right) \\ v_8 &= b_0 \pm b_1\sqrt{\frac{m^2 c_2}{a_2(m^2+1)}}\,\mathrm{sn}\left(\sqrt{-\frac{c_2}{m^2+1}}\xi\right) \\ w_8 &= d_0 \pm 2b_0 b_1\sqrt{\frac{m^2 c_2}{a_2(m^2+1)}}\,\mathrm{sn}\left(\sqrt{-\frac{c_2}{m^2+1}}\xi\right) \qquad c_2 < 0 \end{aligned} \tag{3.13}$$

where $\xi = x + \frac{a_2 c_2 - 6b_1^2}{4a_2}t$. As $m \to 1$, the Jacobi periodic solutions (3.11) and (3.12) degenerate to the soliton solution (3.9), and (3.13) degenerates to (3.10). Taking $b_0 = d_0 = 0$, above solutions directly reduce to the solutions of the Hirota–Satsuma coupled equation.

Similar to example 1, we can deal with other equations and list the results as follows.

Example 2. The KdV–MKdV equation [38, 39]

$$u_t + (\alpha + \beta u)uu_x + u_{xxx} = 0$$

admits two soliton solutions

$$u_1 = -\frac{\alpha}{2\beta} + \frac{\sqrt{6(\alpha^2 - 4\beta c)}}{2\beta}\,\mathrm{sech}\left(\sqrt{\frac{\alpha^2 - 4\beta c}{4\beta}}\xi\right) \qquad \alpha^2 - 4\beta c > 0$$

$$u_2 = -\frac{\alpha}{2\beta} \pm \frac{1}{2\beta}\sqrt{\frac{4\beta c - \alpha^2}{2}}\,\tanh\left(\sqrt{\frac{4\beta c - \alpha^2}{8\beta}}\xi\right) \qquad 4\beta c - \alpha^2 > 0$$

and three Jacobi periodic solutions

$$u_3 = -\frac{\alpha}{2\beta} + \frac{m}{2\beta}\sqrt{\frac{6(\alpha^2 - 4\beta c)}{2m^2 - 1}}\,\mathrm{cn}\left(\sqrt{\frac{\alpha^2 - 4\beta c}{4\beta(2m^2-1)}}\xi\right) \qquad \alpha^2 - 4\beta c > 0$$

$$u_4 = -\frac{\alpha}{2\beta} + \frac{1}{2\beta}\sqrt{\frac{6(\alpha^2 - 4\beta c)}{2 - m^2}}\,\mathrm{dn}\left(\sqrt{\frac{\alpha^2 - 4\beta c}{4\beta(2 - m^2)}}\xi\right) \qquad \alpha^2 - 4\beta c > 0$$

$$u_5 = -\frac{\alpha}{2\beta} \pm \frac{m}{2\beta}\sqrt{\frac{4\beta c - \alpha^2}{m^2 + 1}}\,\mathrm{sn}\left(\sqrt{\frac{4\beta c - \alpha^2}{4\beta(m^2 + 1)}}\xi\right) \qquad \alpha^2 - 4\beta c < 0.$$

As $m \to 1$, the Jacobi periodic solutions u_3 and u_4 degenerate to soliton solutions u_1, and u_5 degenerates to u_2.

Example 3. Ito's fifth-order MKdV equation [18, 40]

$$u_t + \left(6u^5 - 10u^2 u_{xx} - 10uu_x^2 + u_{xxxx}\right)_x = 0$$

admits a soliton solution

$$u_1 = \sqrt{\frac{-c_2}{2}}\tanh\left(\sqrt{\frac{-c_2}{2}}\xi\right) \qquad c_2 < 0$$

and a Jacobi periodic solution

$$u_2 = \sqrt{\frac{-c_2 m^2}{m^2 + 1}}\,\mathrm{sn}\left(\sqrt{\frac{-c_2}{m^2 + 1}}\xi\right) \qquad c_2 < 0$$

where $\xi = x - \left(2a_1^2 c_0 + c_2^2\right)t$. The Jacobi periodic solution u_2 degenerates to the soliton solution u_1 when $m \to 1$.

Example 4. Considering the Broer–Kaup equation [41, 42]

$$u_t + uu_x + v_x = 0 \qquad v_t + u_x + (uv)_x + u_{xxx} = 0$$

we obtain a soliton solution

$$u_1 = -c \pm 2\sqrt{-2c_2}\tanh\left(\sqrt{-\frac{c_2}{2}}\xi\right)$$

$$v_1 = -1 - c_2\,\mathrm{sech}^2\left(\sqrt{-\frac{c_2}{2}}\xi\right) \qquad c < 0$$

and a Jacobi periodic solution

$$u_2 = -c \pm 2\sqrt{-\frac{c_2 m^2}{m^2 + 1}}\,\mathrm{sn}\left(\sqrt{-\frac{c_2}{m^2 + 1}}\xi\right)$$

$$v_2 = -1 - c_2 + \frac{2c_2 m^2}{m^2 + 1}\,\mathrm{sn}^2\left(\sqrt{-\frac{c_2}{m^2 + 1}}\xi\right) \qquad c < 0$$

where $\xi = x + ct$. The Jacobi solution (u_2, v_2) degenerates to the soliton solution (u_1, v_1) as $m \to 1$.

Example 5. The (2 + 1)-dimensional dispersive long wave equation [14, 43]

$$u_{yt} + u_{xx} + \tfrac{1}{2}(u^2)_{xy} = 0 \qquad v_t + (uv + u + u_{xy})_x = 0$$

admits a soliton solution

$$u_1 = -d \pm \sqrt{-2c_2}\tanh\left(\sqrt{-\frac{c_2}{2}}\xi\right)$$

$$v_1 = -(1 + cc_2) - cc_2\tanh^2\left(\sqrt{-\frac{c_2}{2}}\xi\right) \qquad c_2 < 0$$

and a Jacobi periodic solution

$$u_2 = -d \pm 2\sqrt{-\frac{c_2m^2}{m^2+1}}\,\mathrm{sn}\left(\sqrt{-\frac{c_2}{m^2+1}}\xi\right)$$
$$v_2 = -(1+cc_2) - \frac{2cc_2m^2}{m^2+1}\,\mathrm{sn}^2\left(\sqrt{-\frac{c_2}{m^2+1}}\xi\right)$$

where $\xi = x + cy + dt$. The periodic solution (u_2, v_2) degenerates to the soliton solution (u_1, v_1) when $m \to 1$.

In the following, we extend our proposed method to special equations whose solutions require some kinds of pre-possessing techniques.

Example 6. The coupled Schrödinger–KdV equation

$$iu_t = u_{xx} + uv \qquad v_t + 6vv_x + v_{xxx} = (|u|^2)_x \tag{3.14}$$

is known to describe various processes in dusty plasma, such as Langmuir, dust-acoustic wave and electromagnetic waves [44–46]. Here our proposed method is applied to system (3.14) and gives a series of exact solutions.

We introduce the transformation

$$u = \mathrm{e}^{\mathrm{i}\theta}U(\xi) \qquad v = V(\xi) \qquad \theta = px + qt \qquad \xi = x + ct \tag{3.15}$$

where p, q and c are constants.

Substituting (3.15) into (3.14), we find that $c = 2p$ and U, V satisfy the following coupled ordinary differential system:

$$U'' + (q - p^2)U + UV = 0 \qquad 2pV' + 6VV' + V''' - (U^2)' = 0. \tag{3.16}$$

Balancing the highest linear term with nonlinear terms in equation (3.9) allows us to choose the following expansions:

$$U = a_0 + a_1\varphi + a_2\varphi^2 \qquad V = b_0 + b_1\varphi + b_2\varphi^2 \tag{3.17}$$

where φ satisfies equation (2.12).

Substituting (3.17) into (3.16) and using *Mathematica*, we get the following system of algebraic equations:

$$-2p^2a_0 + 2qa_0 + 2a_0b_0 + 4\varepsilon^2a_2c_0 + \varepsilon^2a_1c_1 = 0$$
$$-2p^2a_1 + 2qa_1 + 2a_1b_0 + 2a_0b_1 + 6\varepsilon^2a_2c_1 + 2\varepsilon^2a_1c_2 = 0$$
$$-2p^2a_2 + 2qa_2 + 2a_2b_0 + 2a_1b_1 + 2a_0b_2 + 8\varepsilon^2a_2c_2 + 3\varepsilon^2a_1c_3 = 0$$
$$2a_2b_1 + 2a_1b_2 + 10\varepsilon^2a_2c_3 + 4\varepsilon^2a_1c_4 = 0$$
$$2a_2b_2 + 12\varepsilon^2a_2c_4 = 0$$
$$-2\varepsilon a_0a_1 + 2\varepsilon pb_1 + 6\varepsilon b_0b_1 + 3\varepsilon^3b_2c_1 + \varepsilon^3b_1c_2 = 0$$
$$-2\varepsilon a_1^2 - 4\varepsilon a_0a_2 + 6\varepsilon b_1^2 + 4\varepsilon pb_2 + 12\varepsilon b_0b_2 + 8\varepsilon^3b_2c_2 + 3\varepsilon^3b_1c_3 = 0$$
$$-6\varepsilon a_1a_2 + 18\varepsilon b_1b_2 + 15\varepsilon^3b_2c_3 + 6\varepsilon^3b_1c_4 = 0$$
$$-4\varepsilon a_2^2 + 12\varepsilon b_2^2 + 24\varepsilon^3b_2c_4 = 0.$$

Since $\varepsilon^3 = \varepsilon$, $\varepsilon^2 = 1$, we may eliminate ε from above system. Then solving the system by *Mathematica* gives two sets of solutions

$$c_3 = c_1 = a_1 = b_1 = 0 \qquad c_4 = -\frac{a_2}{6\sqrt{2}} \qquad b_2 = \frac{a_2}{\sqrt{2}} \qquad b_0 = -\frac{1}{3}(p + 6c_2 \pm \delta)$$
$$q = \tfrac{1}{3}(p + 3p^2 \pm \delta) \qquad a_0 = -\sqrt{2}(2c_2 \pm \delta) \qquad \delta = \sqrt{\sqrt{2}a_2c_0 + 4c_2^2} \tag{3.18}$$

with c_0, c_2, p and a_2 being arbitrary constants and

$$c_3 = c_1 = a_0 = b_1 = a_2 = 0 \qquad b_0 = -\frac{1}{3}\left(2p + 4c_2 + \frac{a_1^2}{b_2}\right)$$
$$c_4 = -\frac{1}{2}b_2 \qquad q = \frac{1}{4}\left(2p + 2p^2 - 2c_2 - \frac{a_1^2}{b_2}\right) \tag{3.19}$$

with c_0, c_2, p and $a_1, b_2 \neq 0$ being arbitrary constants.

The various travelling wave solutions of equation (3.14) are discussed as follows:

(A) Similar to the discussion of case A in example 1, (2.13), (2.16) and (3.18) lead to the same kind of soliton solution, namely,

$$u_1 = \mathrm{e}^{\mathrm{i}\theta}[-\sqrt{2}(2c_2 \pm \delta) + 6\sqrt{2}c_2\,\mathrm{sech}^2(\sqrt{c_2}\xi)]$$
$$v_1 = -\tfrac{1}{3}(p + 6c_2 \pm \delta) + 6c_2\,\mathrm{sech}^2(\sqrt{c_2}\xi) \qquad c_2 > 0 \tag{3.20}$$

where $\xi = x + 2pt, \theta = px + \frac{1}{3}(p + 3p^2 \pm \delta)t$.

From (2.18)–(2.20) and (3.18), we obtain Jacobi periodic solutions

$$u_2 = \mathrm{e}^{\mathrm{i}\theta}\left[-\sqrt{2}(2c_2 \pm \delta) + \frac{6\sqrt{2}m^2c_2}{2m^2-1}\,\mathrm{cn}^2\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right)\right]$$
$$v_2 = -\frac{1}{3}(p + 6c_2 \pm \delta) + \frac{6m^2c_2}{2m^2-1}\,\mathrm{cn}^2\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right) \qquad c_2 > 0 \tag{3.21}$$

where $\xi = x + 2pt, \theta = px + \frac{1}{3}(p + 3p^2 \pm \delta)t$. The Jacobi periodic solution (3.21) degenerates to the soliton solution (3.20) as $m \to 1$.

(B) From (2.13), (2.16) and (3.19), we get two soliton solutions

$$u_3 = a_1\sqrt{\frac{2c_2}{b_2}}\,\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{sech}(\sqrt{c_2}\xi)$$
$$v_3 = -\frac{1}{3}\left(2p + 4c_2 + \frac{a_1^2}{b_2}\right) + 2c_2\,\mathrm{sech}^2(\sqrt{c_2}\xi) \qquad c_2 > 0 \tag{3.22}$$

and

$$u_4 = \pm a_1\,\mathrm{e}^{\mathrm{i}\theta}\sqrt{\frac{c_2}{b_2}}\tanh\left(\sqrt{-\frac{c_2}{2}}\xi\right)$$
$$v_4 = -\frac{1}{3}\left(2p + 4c_2 + \frac{a_1^2}{b_2}\right) + c_2\tanh^2\left(\sqrt{-\frac{c_2}{2}}\xi\right) \qquad c_2 < 0 \tag{3.23}$$

where $\xi = x + 2pt, \theta = px + \frac{1}{4}\left(2p + 2p^2 - 2c_2 - \frac{a_1^2}{b_2}\right)t$.

From (2.18)–(2.20) and (3.19), we get three Jacobi doubly periodic solutions as follows:

$$u_5 = a_1\,\mathrm{e}^{\mathrm{i}\theta}\sqrt{\frac{2m^2c_2}{b_2(2m^2-1)}}\,\mathrm{cn}\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right)$$
$$v_5 = -\frac{1}{3}\left(2p + 4c_2 + \frac{a_1^2}{b_2}\right) + \frac{2m^2c_2}{2m^2-1}\,\mathrm{cn}^2\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right) \qquad c_2 > 0 \tag{3.24}$$

$$u_6 = a_1\,\mathrm{e}^{\mathrm{i}\theta}\sqrt{\frac{2c_2}{b_2(2-m^2)}}\,\mathrm{dn}\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right)$$
$$v_6 = -\frac{1}{3}\left(2p + 4c_2 + \frac{a_1^2}{b_2}\right) + \frac{2c_2}{2-m^2}\,\mathrm{dn}^2\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right) \qquad c_2 > 0 \tag{3.25}$$

and

$$u_7 = \pm a_1\,\mathrm{e}^{\mathrm{i}\theta}\sqrt{\frac{2m^2c_2}{b_2(m^2+1)}}\,\mathrm{sn}\left(\sqrt{-\frac{c_2}{m^2+1}}\xi\right)$$
$$v_7 = -\frac{1}{3}\left(2p+4c_2+\frac{a_1^2}{b_2}\right)+\frac{2m^2c_2}{m^2+1}\,\mathrm{sn}^2\left(\sqrt{-\frac{c_2}{m^2+1}}\xi\right) \qquad c_2<0 \tag{3.26}$$

where $\xi = x+2pt, \theta = px + \frac{1}{4}\big(2p+2p^2-2c_2-\frac{a_1^2}{b_2}\big)t$. As $m \to 1$, the Jacobi periodic solutions (3.24) and (3.25) degenerate to the soliton solution (3.22), and (3.26) degenerates to (3.23).

In a similar way to example 6, we can deal with other special-type nonlinear equations and list the corresponding results as follows.

Example 7. The Hirota equation reads [47, 48]

$$\mathrm{i}u_t + u_{xx} + 2|u|^2u + \mathrm{i}\alpha u_{xxx} + 6\,\mathrm{i}\alpha|u|^2u_x = 0$$

which is the standard Schrödinger equation in the case when $\alpha = 0$.

The Hirota equation admits two soliton solutions

$$u_1 = \sqrt{c_2}\,\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{sech}(\sqrt{c_2}\xi) \qquad c_2>0$$
$$u_2 = \pm\,\mathrm{e}^{\mathrm{i}\theta}\sqrt{-\frac{c_2}{2}}\tanh\left(\sqrt{-\frac{c_2}{2}}\xi\right) \qquad c_2<0$$

and three Jacobi doubly periodic wave solutions

$$u_3 = \mathrm{e}^{\mathrm{i}\theta}\sqrt{\frac{c_2m^2}{2m^2-1}}\,\mathrm{cn}\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right) \qquad c_2>0$$
$$u_4 = \mathrm{e}^{\mathrm{i}\theta}\sqrt{\frac{c_2}{2-m^2}}\,\mathrm{dn}\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right) \qquad c_2>0$$
$$u_5 = \pm\mathrm{e}^{\mathrm{i}\theta}\sqrt{-\frac{c_2m^2}{m^2+1}}\,\mathrm{sn}\left(\sqrt{-\frac{c_2}{m^2+1}}\xi\right) \qquad c_2<0$$

where $\theta = \frac{1}{3\alpha}x - \frac{2}{27\alpha^2}t, \xi = x - \frac{1}{3\alpha}\big(1+3\alpha^2c_2\big)t$. As $m \to 1$, the Jacobi periodic solutions u_3 and u_4 all degenerate to the same soliton solution u_1, and u_5 degenerates to u_2. Now as an illustrative sample, we draw the plots for these solutions of the Hirota equation. The properties of soliton solutions u_1 and u_2 are shown in figures 1 and 2. The properties of the Jacobi doubly periodic solutions u_3, u_4 and u_5 are shown in figures 3–5, respectively.

Example 8. Nizhnik–Novikov–Veselov equation [49, 50]

$$u_t + u_{xxx} + u_{yyy} + 3\left(u\partial_y^{-1}u_x\right)_x + 3\left(u\partial_x^{-1}u_y\right)_y = 0$$

admits a soliton solution

$$u_1 = a_0 - \frac{cc_2(1+c)(c^2-c+1)}{c^2+1}\,\mathrm{sech}^2\left(\frac{\sqrt{c_2}}{2}\xi\right) \qquad c_2>0$$

and a Weierstrass periodic solution

$$u_2 = a_0 + \wp\left(\frac{1}{2}\sqrt{\frac{2a_1(1+c^2)}{c(1+c)(c^2-c+1)}}\xi, g_2, g_3\right)$$

where $g_2 = -4c_1/c_3, g_3 - 4c_0/c_3, \xi = x + cy - \big(\frac{6a_0}{c} + 6ca_0 + c_2 + c^3c_2\big)t$.

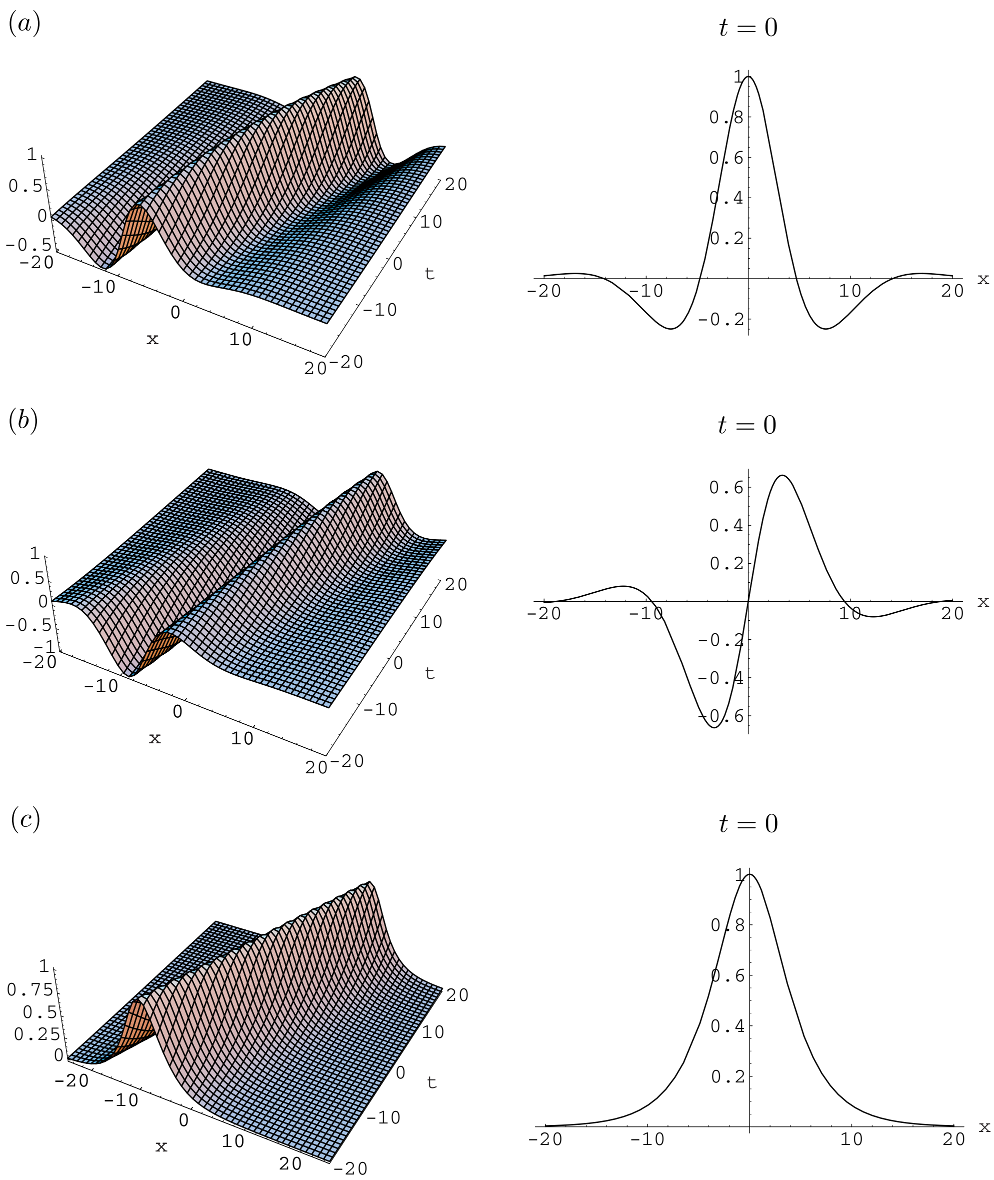

Figure 1. The soliton solution u_1 and its position at $t = 0$, where parameters $\alpha = 1, c_2 = 0.1$. (*a*) The real part, (*b*) the imaginary part, (*c*) the modulus.

Example 9. The (2 + 1)-dimensional coupled Davey–Stewartson equation [1, 4, 51]

$$\mathrm{i}u_t + u_{xx} - u_{yy} - 2|u|^2u - 2uv = 0 \qquad v_{xx} + v_{yy} + 2(|u|^2)_{xx} = 0$$

admits two soliton solutions

$$u_1 = \mathrm{e}^{\mathrm{i}\theta}\sqrt{c_2(1+c^2)}\,\mathrm{sech}(\sqrt{c_2}\xi)$$
$$v_1 = \tfrac{1}{2}(q^2 - p^2 - k + c_2 - c^2c_2) - \tfrac{1}{2}c_2\,\mathrm{sech}^2(\sqrt{c_2}\xi) \qquad c_2 > 0$$

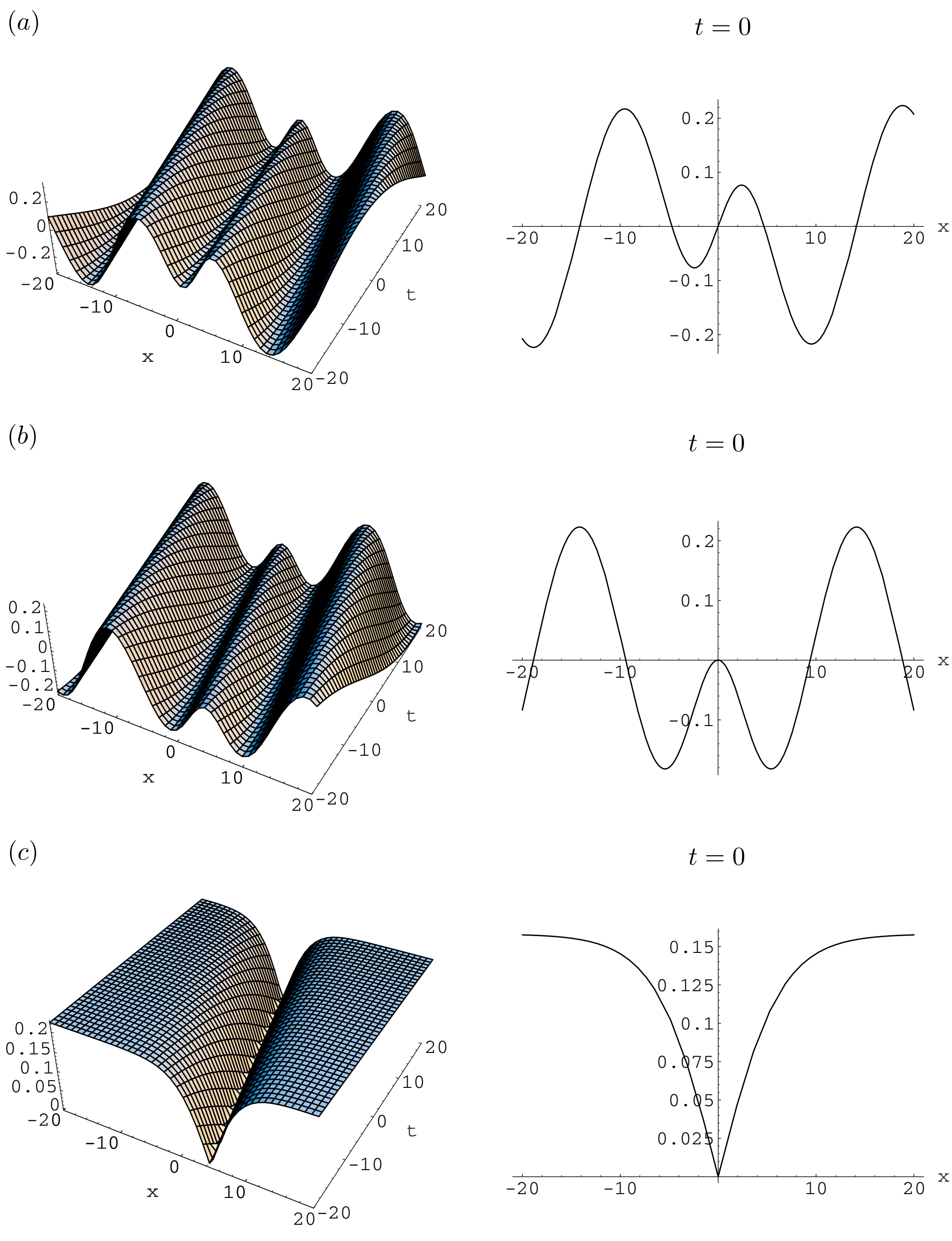

Figure 2. The soliton solution u_2 and its position at $t = 0$, where parameters $\alpha = 1$, $c_2 = -0.1$. (*a*) The real part, (*b*) the imaginary part, (*c*) the modulus.

$$u_2 = \pm \mathrm{e}^{\mathrm{i}\theta}\sqrt{-\frac{c_2(1+c^2)}{2}}\tanh\left(\sqrt{\frac{c_2}{2}}\xi\right)$$

$$v_2 = \frac{1}{2}(q^2 - p^2 - k + c_2 - c^2c_2) + \frac{c_2(1+c^2)}{2}\tanh^2\left(-\sqrt{\frac{c_2}{2}}\xi\right) \qquad c_2 < 0$$

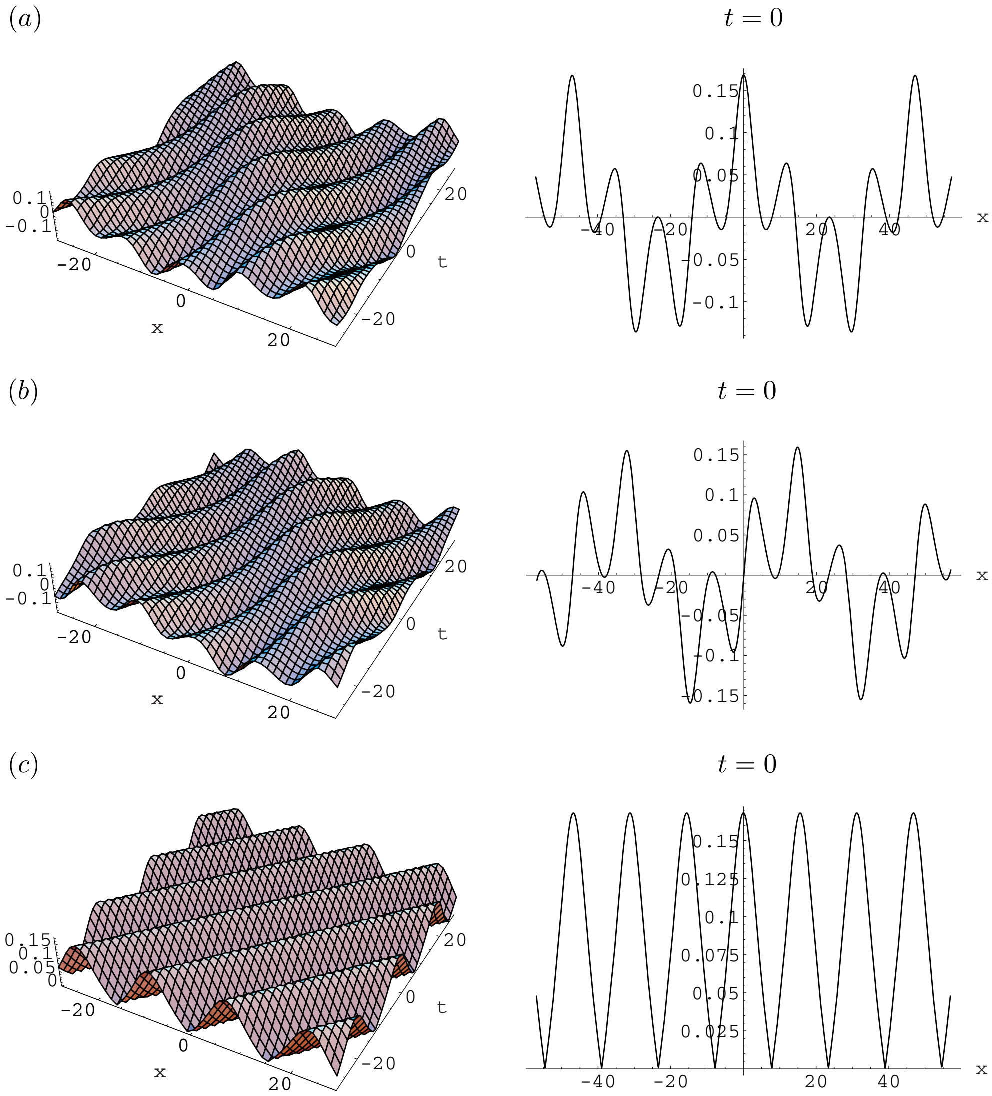

Figure 3. The Jacobi doubly periodic solution u_3 and its position at $t = 0$, where parameters $\alpha = 0.5$, $c_2 = 0.3$. (*a*) The real part, (*b*) the imaginary part, (*c*) the modulus.

and three Jacobi doubly periodic solutions

$$u_3 = e^{i\theta}\sqrt{\frac{2m^2(1+c^2)}{2m^2-1}}\,\mathrm{cn}\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right)$$

$$v_3 = \frac{1}{2}(q^2 - p^2 - k + c_2 - c^2c_2) + \frac{m^2c_2}{2(2m^2-1)}\,\mathrm{cn}^2\left(\sqrt{\frac{c_2}{2m^2-1}}\xi\right) \qquad c_2 > 0$$

$$u_4 = e^{i\theta}\sqrt{\frac{2(1+c^2)}{2-m^2}}\,\mathrm{dn}\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right)$$

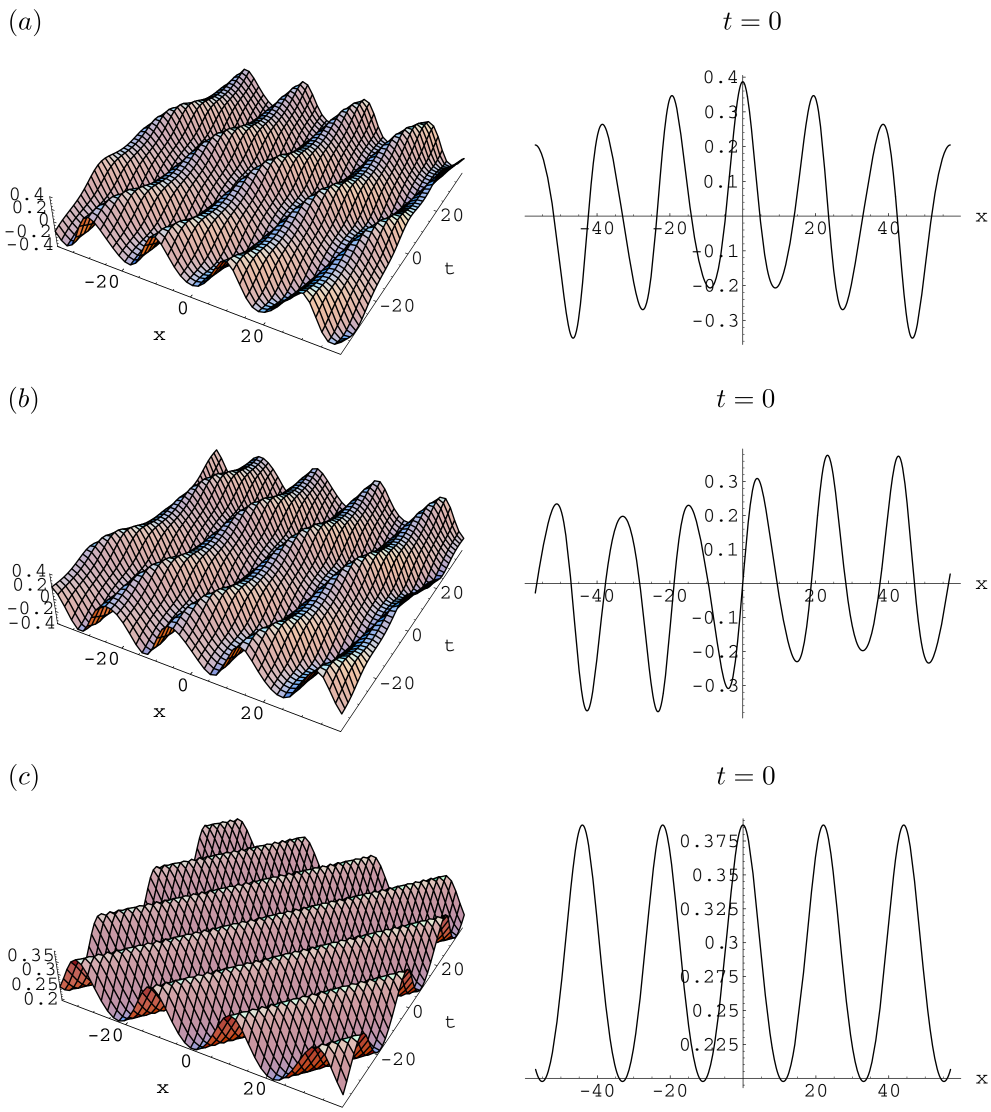

Figure 4. The Jacobi doubly periodic solution u_4 and its position at $t = 0$, where $\alpha = 0.5$, $c_2 = 0.3$. (*a*) The real part, (*b*) the imaginary part, (*c*) the modulus.

$$v_4 = \frac{1}{2}(q^2 - p^2 - k + c_2 - c^2 c_2) + \frac{c_2}{2(2-m^2)} \operatorname{dn}^2\left(\sqrt{\frac{c_2}{2-m^2}}\xi\right) \qquad c_2 > 0$$

$$u_5 = \pm \mathrm{e}^{\mathrm{i}\theta} \sqrt{\frac{m^2(1+c^2)}{m^2+1}} \operatorname{sn}\left(\sqrt{-\frac{c_2}{m^2+1}}\xi\right)$$

$$v_5 = \frac{1}{2}(q^2 - p^2 - k + c_2 - c^2 c_2) + \frac{m^2 c_2 (1+c^2)}{m^2+1} \operatorname{sn}^2\left(\sqrt{-\frac{c_2}{m^2+1}}\xi\right) \qquad c_2 < 0$$

where $\theta = px + qy + kt$, $\xi = x + cy - 2(p - qc)t$. As $m \to 1$, Jacobi periodic solutions (u_3, v_3) and (u_4, v_4) degenerate to soliton solutions (u_1, v_1) and (u_2, v_2), respectively.

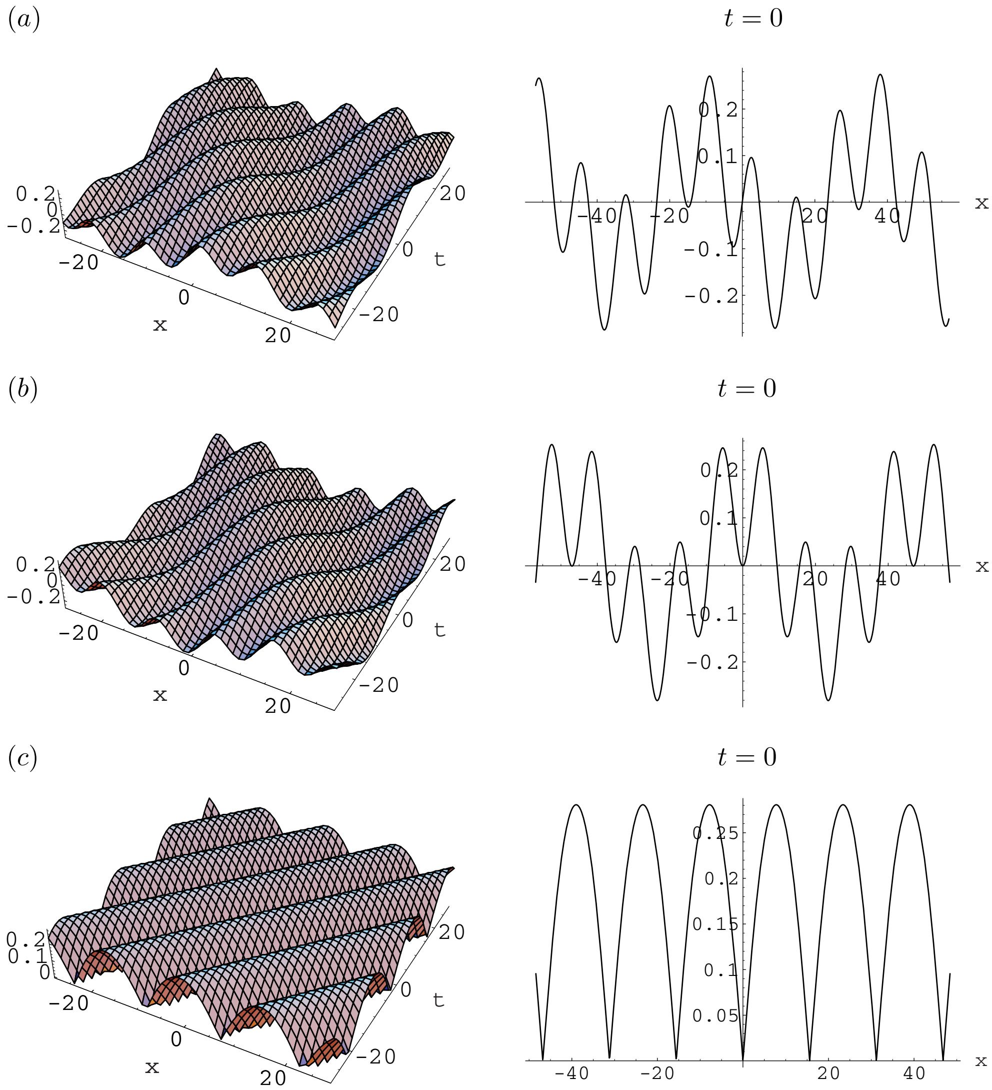

Figure 5. The Jacobi doubly periodic solution u_5 and its position at $t = 0$, where $\alpha = 1$, $c_2 = -0.5$. (*a*) The real part, (*b*) the imaginary part, (*c*) the modulus.

4. Conclusion

In summary, we have proposed a unified algebraic method with symbolic computation, which greatly exceeds the applicability of the existing tanh method, extended tanh method and Jacobi elliptic function method in obtaining multiple travelling wave solutions of general nonlinear evolution equations. The feature of our proposed method is that, without much extra effort, we circumvent integration to directly get the above series explicit solutions in a uniform way. Another merit is that the method is independent of the integrability of nonlinear equations, so that it can be used to solve both integrable and non-integrable nonlinear equations. Viewed as a special case of partial differential equations, the method readily applies to nonlinear ordinary differential equations.

Except those considered in this paper, the proposed method also is readily applicable to a large variety of other nonlinear equations including classical KdV, MdV, Jaulent–Miodek, BBM, modified BBM, Benjamin Ono, Kawachra, variant Boussinesq, Schrödinger, Klein–Gordon, sine-Gordon, sinh-Gordon, Dodd–Bullough–Mikhailov, $(2+1)$-dimensional KP, $(2+1)$-dimensional Kaup–Kupershmidt, $(2+1)$-dimensional Gardner, $(3+1)$-dimensional Jimbo–Miwa, coupled KdV, coupled Schrödinger–Boussinesq and coupled Ito equations, etc. In addition, according to our proposed method the travelling wave solutions of a given nonlinear equation depend on the explicit solvability of (2.8) with its coefficients c, a_i, c_j satisfying a system of algebraic equations. In this paper, we have only investigated a special case when $r = 4$. The proposed method can be extended to the case when $r > 4$. The details for these cases will be investigated in our future works.

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